Problem 1: Legendre series

Express the following functions as Legendre series (as a sum of Legendre polynomials)

(a) $3\,x^2 - 4$

(b) $2\,x^4 + 6\,x^2 + 2$

(c) $-4\,\cos^5\theta$ (In this case, your Legendre Polynomials should be functions of $\cos\theta$.)

You may determine the coefficients in the series algebraically or by using the orthogonality properties of the Legendre polynomials (in other words, using Fourier's trick). In either case you must show all of your work to receive credit.

Problem 2: Another conducting pipe

A very long metal pipe with a rectangular cross section runs parallel to the z axis (we discussed a similar example in class). The two sides of the pipe at $x = 0$ and $x = a$ are grounded, so $V(0, y) = V(a, y) = 0$. The other two sides of the pipe, at $y = 0$ and $y = b$, are held at opposite, constant potentials: $V(x, 0) = -V_0$ and $V(x, b) = V_0$. Find the potential $V(x, y)$ inside the pipe. You may assume that the pipe is so long that the potential does not depend on z.

Problem 3: Potential inside a conducting cube

A cubical box with sides of length L consists of six metal plates. Five sides of the box – the plates at $x = 0$, $x = L$, $y = 0$, $y = L$, and $z = 0$ – are grounded. The top of the box (at $z = L$) is made of a separate sheet of metal, insulated from the others, and held at a constant potential V_0. Find the potential inside the box.

HINT: Keep in mind that the answer here depends on x, y, and z. So you will need to work out a separable solution in all three variables, impose as many of the boundary conditions as possible, write down an appropriate sum of the resulting separable solutions, and then apply any remaining boundary conditions to identify the unique solution. You should be able to impose all five homogeneous boundary conditions. This will give you building blocks $\mathcal{F}_{n,m}(x, y, z)$ that depend on two independent integers n and m. Write the potential as a double sum over all possible building blocks:

$$V(x, y, z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \mathcal{C}_{n,m}\,\mathcal{F}_{n,m}(x, y, z)\ .$$

Now set $z = L$

$$V(x, y, L) = V_0 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \mathcal{C}_{n,m}\,\mathcal{F}_{n,m}(x, y, L)$$

and apply Fourier's trick twice to figure out the coefficients $\mathcal{C}_{n,m}$ for the various building blocks.

Problem 4: Potential due to a spherical surface charge

The potential at the surface of an insulating sphere (radius R) is given by

$$V(R, \theta) = V_0 \cos 3\theta$$

where V_0 is a constant. Use separation of variables to find the potential inside the sphere ($r \leq R$) and outside the sphere ($r \geq R$), and then use your answer to determine the surface charge density $\sigma(\theta)$ on the sphere. Assume there is no charge inside or outside the sphere.

HINT: Someone usually tries to solve this problem by using Legendre polynomials $P_\ell(\cos 3\theta)$ instead of $P_\ell(\cos\theta)$. That's a good idea! Unfortunately, it doesn't work. You can check for yourself that $r^\ell P_\ell(\cos 3\theta)$ and $r^{-(\ell+1)} P_\ell(\cos 3\theta)$ are not solutions of Laplace's equation. Instead, you should use trig identities to express $\cos 3\theta$ as a polynomial in powers $\cos\theta$. You should get a term cubic in $\cos\theta$ and a term linear in $\cos\theta$.

*Problem 5: Opposite charges on opposite hemispheres

A spherical shell of radius R carries a uniform surface charge σ_0 on the "northern" hemisphere ($0 \leq \theta < \pi/2$), and a uniform surface charge $-\sigma_0$ on the "southern" hemisphere ($\pi/2 \leq \theta < \pi$). Find the first five terms (starting with $\ell = 0$) in the Legendre expansion of the potential inside the sphere ($r \leq R$) and outside the sphere ($r \geq R$). That is, calculate or justify the coefficients in the Legendre series for the potential for $\ell \leq 4$. Remember that you have different coefficients inside and outside the sphere!